GENERALIZED INTEGRAL TRANSFORM METHOD FOR THE SOLUTION OF THE HEAT-CONDUCTION EQUATION IN A REGION WITH MOVING BOUNDARIES

É. M. Kartashov

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We develop a method of integral transforms for the solution of the heat-conduction equation in a region with moving boundaries. The method leads to analytical solutions of thermal problems of both classical and Stefan type in new functional forms.

The method of integral transforms, mathematically equivalent to the method of characteristic functions and characteristic values, has received broad application in the solution of the heat-conduction equation in finite (and infinite) regions of classical type [1]. It makes it possible to obtain analytical solutions of thermal problems in the standard way and to then improve the convergence of these solutions [2]. Indeed, let T(M, t) be the solution of a thermal problem in the region $\Omega = \{(M, t): M \in G,\ t > 0\}$:

$$\partial T/\partial t = a\Delta T(M,\ t) + F(M,\ t),\ M \in G,\ t > 0; \tag{1}$$

$$T(M,\ 0) = \Phi_0(M),\ M \in G; \tag{2}$$

$$\beta_1 \partial T(M,\ t)/\partial n - \beta_2 T(M,\ t) = -\varphi(M,\ t),\ M \in S,\ t > 0. \tag{3}$$

Here S is a piecewise-smooth surface bounding the region G; n is the exterior normal to S; $\beta_i \geqq 0$ $(i = 1,\ 2)$, $\beta_1^2 + \beta_2^2 > 0$. The boundary functions in Eqs. (1)-(3) belong to the class of functions [3]: $\Phi_0(M) \in C^1(\bar{G})$; $\varphi(M,\ t) \in C^0(S \times t \geqq 0)$; $F(M,\ t) \in C^0(\Omega)$. When the compatibility conditions $\{\beta_1 \partial\Phi_0(M) - \beta_2\Phi_0(M) = -\varphi(M,\ 0)\}_{M \in S}$ are satisfied (this may not be so in particular cases), the desired solution is a function of the class

$$T(M,\ t) \in C^2(\Omega) \cap C^0(\bar{\Omega});\ \mathrm{grad}_M T(M,\ t) \in C^0(\bar{\Omega}). \tag{4}$$

It is not difficult to find the general solution of the problem (1)-(3) if one first finds the characteristic functions $\Psi_n(M,\ \gamma_n)$ and the characteristic values γ_n^2 of the corresponding homogeneous problem

$$\Delta\Psi(M,\ \gamma) + \gamma^2\Psi = 0,\ M \in G; \tag{5}$$

$$\beta_1 \partial\Psi(M,\ \gamma)/\partial n - \beta_2\Psi(M,\ \gamma) = 0,\ M \in S. \tag{6}$$

To this end we introduce an integral transformation of the form

$$L(T) = \bar{T}(\gamma_n,\ t) = \iiint_G T(M,\ t)\,\Psi_n(M,\ \gamma_n)\,dV_M \tag{7}$$

with the inversion formula

$$T(M,\ t) = \sum_{n=1}^{\infty} \frac{\Psi_n(M,\ \gamma_n)}{\|\Psi_n\|^2}\bar{T}(\gamma_n,\ t);\ \|\Psi_n\|^2 = \iiint_G \Psi_n^2(M,\ \gamma_n)\,dV_M, \tag{8}$$

which results from the possibility of expanding the desired solution T(M, t) in a Fourier series in terms of the system of characteristic functions $\{\Psi_n(M,\ \gamma_n)\}$. Transform (7) yields the following representation for the operator $\Delta T(M,\ t)$:

$$L(\Delta T(M,\ t)) = -\gamma_n^2\bar{T}(\gamma_n,\ t) + \iint_S [\Psi_n(M,\ \gamma_n)\,\partial T(M,\ t)/\partial n - T(M,\ t)\,\partial\Psi_n(M,\ \gamma_n)/\partial n]_{M \in S}\,d\sigma, \tag{9}$$

where boundary conditions of an arbitrary kind may be used in the Eqs. (3) and (6). Hence, applying relations (7) and (8) to problem (1)-(3), we find its analytical solution in the form

$$T(M,\ t) = \sum_{n=1}^{\infty} \frac{\Psi_n(M,\ \gamma_n)}{\|\Psi_n\|^2} \exp[-(\sqrt{a}\,\gamma_n)^2 t] \iiint_G \Phi_0(M)\,\Psi_n(M,\ \gamma_n)\,dV_M +$$

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$$+a\sum_{n=1}^{\infty}\frac{\Psi_n(M,\gamma_n)}{\|\Psi_n\|^2}\int_0^t\iint_S[(-1/\beta_1)\varphi(M,\tau)\Psi_n(M,\gamma_n)]_{M\in S}\times \tag{10}$$

$$\times\exp[-(\sqrt{a}\gamma_n)^2(t-\tau)]\,d\tau d\sigma+\sum_{n=1}^{\infty}\frac{\Psi_n(M,\gamma_n)}{\|\Psi_n\|^2}\int_0^t\iiint_G\exp[-(\sqrt{a}\gamma_n)^2(t-\tau)]F(M,\tau)\Psi_n(M,\gamma_n)\,d\tau dV_M.$$

However, this does not exhaust the problem of constructing the solution (10). Attention is repeatedly directed in the literature to the slow convergence of the series (10) in a neighborhood of boundary points of the region G, as a result of which difficulties arise in the treatment of these series [4]. There is the additional problem of improving the convergence of these series up to the boundary of the domain G. To this end:

1. We construct a quasistatic solution of the problem

$$\Delta\Theta(M,t)=0,\ M\in G;\ \beta_1\partial\Theta(M,t)-\beta_2\Theta(M,t)=-\varphi(M,t),\ M\in S \tag{11}$$

(for $\beta_2 \neq 0$ [2]), applying the integral transformation (7), (8):

$$\Theta(M,t)=\sum_{n=1}^{\infty}\frac{\Psi_n(M,\gamma_n)}{\gamma_n^2\|\Psi_n\|^2}\iint_S[(-1/\beta_1)\varphi(M,t)\Psi_n(M,\gamma_n)]_{M\in S}d\sigma. \tag{12}$$

Simultaneously, we find for $\Theta(M, t)$ a corresponding expression in closed form (for cases with one spatial variable this is always possible; in the contrary case we can apply the product of solutions method [3]).

2. We now subtract the series (12) from the right side of equation (10) and at the same time add $\Theta(M, t)$ in closed form. This yields an improved solution with rapid convergence of the series everywhere in $\tilde{G} = G + S$:

$$\begin{aligned}T(M,t)&=\Theta(M,t)+\sum_{n=1}^{\infty}\frac{\Psi_n(M,\gamma_n)}{\|\Psi_n\|^2}\exp[-(\sqrt{a}\gamma_n)^2t]\times\\&\times\iiint_G\Phi_0(M)\Psi_n(M,\gamma_n)\,dV_M+\sum_{n=1}^{\infty}\frac{\Psi_n(M,\gamma_n)}{\|\Psi_n\|^2}\Big\{a\int_0^t\iint_S\exp[-(\sqrt{a}\gamma_n)^2\times\\&\times(t-\tau)][(-1/\beta_1)\varphi(M,\tau)\Psi_n(M,\gamma_n)]_{M\in S}d\tau d\sigma-(1/\gamma_n)^2\iint_S[(1/\beta_1)\varphi(M,t)\times\\&\times\Psi_n(M,\gamma_n)]_{M\in S}d\sigma+\int_0^t\iiint_G\exp[-(\sqrt{a}\gamma_n)^2(t-\tau)]\,F(M,\tau)\Psi_n(M,\gamma_n)\,d\tau dV_M\Big\}.\end{aligned} \tag{13}$$

Thus, the application of the method of integral transforms in solving thermal problems of the type (1)-(3) in classical regions with nonhomogeneities in the boundary conditions must lead to two basic results: first, that of finding the solution (10) and its improvement to that of the form (13). The second part of the approach, the practical calculation of temperature fields in finite regions, has as yet not received adequate development, a circumstance which leads in many cases to an essential loss in accuracy in the numerical results when handling the series (10).

Let us assume now that in the Eqs. (1)-(3) the region G_t of variation of the spatial variables is bounded by the surface S_t, moving so that $\Omega_t = \{M \in G_t,\ t > 0\}$ is a noncylindrical region. For definiteness we put $\Omega_t = \{0 < x < y(t),\ t > 0\}$, where $y(t)$ is a continuously differentiable function of a general form with $y(0) = y_0 \geq 0$. From the mathematical point of view the boundary value problem for the heat conduction equation

$$\partial T/\partial t=a\partial^2T/\partial x^2+F(x,t),\ 0<x<y(t),\ t>0, \tag{14}$$

in a domain with a moving boundary is fundamentally different from classical problems (1)-(3). Due to the dependence of the characteristic size of the region on the time, we cannot apply the approach (7), (8) to this type of problem in the general case since, apart from the classical-methods framework (based in the final analysis on the possibilitiy of considering the homogeneous problem (5), (6)), there is no way to adjust the solution of Eq. (14) to the motion of the boundary of the region of heat transfer. This adjustment problem requires the development of new approaches. In connection with the case (14) the formulation of such an approach depends essentially on the nature of the relation $y_0 \geq 0$. If $y_0 = 0$, i.e., if the domain of definition of Eq. (14) for $t = 0$ is concentrated at a point, we say that the domain $\bar{\Omega}_t = \{0 \leq x \leq y(t),\ t \geq 0\}$ is degenerate. The method of differential series, developed in [3] for noncylindrical regions, leads to the following sufficiently general expression for the function T(x, t), satisfying in Ω_t the equation (14):

$$T(x,\ t)=T(y(t),\ t)+\sum_{n=1}^{\infty}\frac{1}{a^n(2n)!}\frac{\partial^{n-1}}{\partial t^{n-1}}\left\{[y(t)-x]^{2n}\frac{d}{dt}T(y(t),\ t)\right\}-$$
$$-\sum_{n=0}^{\infty}\frac{1}{a^n(2n+1)!}\frac{\partial^n}{\partial t^n}\{[y(t)-x]^{2n+1}(\partial T/\partial x)_{x=y(t)}\}-\sum_{n=0}^{\infty}\frac{1}{a^{n+1}(2n+1)!}\frac{\partial^n}{\partial t^n}\int_x^{y(t)}(\xi-x)^{2n+1}F(\xi,\ t)\,d\xi, \tag{15}$$

which allows one to construct an analytical solution of a thermal problem in Ω_t for the degenerate case. However, the situation becomes sharply more involved when $y(0) = y_0 > 0$, i.e., when the domain Ω_t is nondegenerate and the approach (15) becomes ineffective (as so also the other approaches systematized in the survey [5]). The latter case can be studied on the basis of the generalized integral transform presented here for noncylindrical domains $x \in [0, y(t)]$, $t > 0$, the fundamentals of which were formulated in [6, 7] in the solution of a particular Stefan problem.

Let

$$\bar{T}(p,\ t)=\int_0^{y(t)}T(x,\ t)\,\mathrm{sh}(x\sqrt{p})\,dx, \tag{16}$$

where $p = \sigma + i\omega$ is a complex number with $\mathrm{Re}\ p \geq \beta > 0$, $|\arg\sqrt{p}| < \pi/4$ is the integral transform of the function T(x, t). With the aid of the transformation (16) we obtain, first of all, a general formula for T(x, t) satisfying Eq. (14) in a nondegenerate region. In the transform space (16) the solution of the transformed equation (14) has the form

$$\bar{T}(p,\ t)=\exp(apt)\int_0^{y_0}T(x,\ 0)\,\mathrm{sh}(x\sqrt{p})\,dx+\int_0^t\exp[ap(t-\tau)]\times$$
$$\times\{a\,\mathrm{sh}(y(\tau)\sqrt{p})(\partial T/\partial x)_{x=y(\tau)}+[(dy(\tau)/d\tau)\,\mathrm{sh}(y(\tau)\sqrt{p})-$$
$$-a\sqrt{p}\,\mathrm{ch}[y(\tau)\sqrt{p}]\,T(y(\tau),\ \tau)+a\sqrt{p}\,T(0,\ \tau)+\bar{F}(p,\ \tau)\}\,d\tau; \tag{17}$$
$$\bar{F}(p,\ t)=\int_0^{y(t)}F(x,\ t)\,\mathrm{sh}(x\sqrt{p})\,dx.$$

In relation (17) there appear boundary values of the function T(x, t) and its derivative. The inversion formula for the transform (16) depends for its formulation on the type of boundary conditions. In the case of boundary conditions of the first kind, we seek an inversion formula in the form of a Fourier-type series corresponding to the classical solution of the first boundary-value problem for Eq. (14):

$$T(x,\ t)=\sum_{n=1}^{\infty}a_n(t)\exp\left\{-\left[\frac{\sqrt{a}\,n\pi}{y(t)}\right]^2t\right\}\sin\frac{n\pi x}{y(t)}. \tag{18}$$

We note that in the case of boundary conditions of the second kind for x = 0 and of the first or second kinds for x = y(t), as the kernel of the transform (16) we assume the function $\mathrm{ch}(x\sqrt{p})$. The inversion formula is then constructed according to the classical solution.

To relation (18) we apply the transform (16); this gives

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}na_n(t)\exp[-(\sqrt{a}\,n\pi/y(t))^2t]}{p+n^2\pi^2/y^2(t)}=\frac{\bar{T}(p,\ t)\,y(t)}{\pi\,\mathrm{sh}[y(t)\sqrt{p}]}, \tag{19}$$

where $\bar{T}(p, t)$ is the function (17). Expression (19) allows us to calculate the unknown coefficients $a_n(t)$. To this end, we integrate both sides of Eq. (19) along the contours γ_1, γ_2, ..., γ_n, ... The contour γ_n consists of the vertical $\sigma > 0$, the two horizontal lines $\omega_n = \pm(2n^2 + 2n + 1)[\pi^2/2y^2(t)]$, and the semicircle with center at the origin and of radius $R_n = (2n^2 + 2n + 1)[\pi^2/2y^2(t)]$. In sequence, we find:

$$a_1(t)=\frac{y(t)}{\pi}\exp\left[\left(\frac{\sqrt{a}\,\pi}{y(t)}\right)^2t\right]\frac{1}{2\pi i}\int_{\gamma_1}\frac{\bar{T}(p,\ t)\,dp}{\mathrm{sh}[y(t)\sqrt{p}]}; \tag{20}$$

$$a_n(t)=\frac{(-1)^{n+1}y(t)}{n\pi}\exp\left[\left(\frac{\sqrt{a}\,n\pi}{y(t)}\right)^2 t\right]\left\{\frac{1}{2\pi i}\int\limits_{\gamma_n}\frac{\overline{T}(p,\,t)\,dp}{\operatorname{sh}[y(t)\sqrt{p}]}-\frac{1}{2\pi i}\int\limits_{\gamma_{n-1}}\frac{\overline{T}(p,\,t)\,dp}{\operatorname{sh}[y(t)\sqrt{p}]}\right\},\ (n\geqslant 2). \quad (21)$$

Evaluating the contour integrals in Eqs. (20) and (21), we arrive at the desired coefficients $a_n(t)$ in the following form:

$$\begin{aligned} a_n(t)=\frac{2}{y(t)}&\left\{\int\limits_0^{y_0}T(x,\,0)\sin\frac{n\pi x}{y(t)}\,dx+\int\limits_0^t\exp\left[\left(\frac{\sqrt{a}\,n\pi}{y(t)}\right)^2\tau\right]\left(\frac{\partial T}{\partial x}\right)_{x=y(\tau)}\times\right.\\ &\times\sin\frac{n\pi y(\tau)}{y(t)}\,d\tau+\frac{n\pi a}{y(t)}\int\limits_0^t T(0,\,\tau)\exp\left[\left(\frac{\sqrt{a}\,n\pi}{y(t)}\right)^2\tau\right]d\tau+\\ &+\int\limits_0^t\left[\frac{dy(\tau)}{d\tau}\sin\frac{n\pi y(\tau)}{y(t)}-\frac{n\pi a}{y(t)}\cos\frac{n\pi y(\tau)}{y(t)}\right]\times\\ &\left.\times\exp\left[\left(\frac{\sqrt{a}\,n\pi}{y(t)}\right)^2\tau\right]T(y(\tau),\,\tau)\,d\tau+\int\limits_0^t\exp\left[\left(\frac{n\pi\sqrt{a}}{y(t)}\right)^2\tau\right]d\tau\int\limits_0^{y(\tau)}F(x,\,\tau)\sin\frac{n\pi x}{y(t)}\,\partial x\right\},\quad n\geqslant 1. \end{aligned} \quad (22)$$

Thus, all the relations necessary for consideration of a series of thermal problems for Eq. (14) in the domain $x\in[0,\ y(t)]$, $t>0$, generalizing the classical cases, have been obtained. Suppose, for example, that Eq. (14) is considered in the classical domain $x\in[0,\ y_0]$, $t\geq 0$ ($y(t)=y_0=\text{const}$, $t\geq 0$), and, moreover, that $T(x,\,0)=f(x)$, $0<x<y_0$; $T(0,\,t)=\varphi_1(t)$, $T(y_0,\,t)=\varphi_2(t)$. From relations (18) and (22) we obtain the well-known classical solution of the first boundary-value problem [3]:

$$\begin{aligned} T(x,\,t)=&\frac{2}{y_0}\sum_{n=1}^{\infty}\exp\left[-\left(\frac{n\pi\sqrt{a}}{y_0}\right)^2 t\right]\sin\frac{n\pi x}{y_0}\int\limits_0^{y_0}f(x)\sin\frac{n\pi x}{y_0}\,dx+\\ &+\frac{2\pi a}{y_0^2}\sum_{n=1}^{\infty}n\sin\frac{n\pi x}{y_0}\int\limits_0^t\exp\left[-\left(\frac{n\pi\sqrt{a}}{y_0}\right)^2(t-\tau)\right][\varphi_1(\tau)-(-1)^n\varphi_2(\tau)]\,d\tau+\frac{2}{y_0}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{y_0}\times\\ &+\int\limits_0^t\int\limits_0^{y_0}\exp\left[-\left(\frac{n\pi\sqrt{a}}{y_0}\right)^2(t-\tau)\right]F(x,\,\tau)\sin\frac{n\pi x}{y_0}\,d\tau dx. \end{aligned} \quad (23)$$

We consider now a number of applications of relations (18) and (22), not available in the literature. We write out the solution of a sufficiently general problem for Eq. (14) with a free boundary, generalizing thereby a well-known Stefan problem:

$$\partial T/\partial t=a\partial^2T/\partial x^2,\ 0<x<y(t),\ t>0; \quad (24)$$

$$T(x,\,0)=\Phi_0(x),\ 0<x<y_0;\ T(0,\,t)=\varphi_1(t),\ t>0; \quad (25)$$

$$T(y(t),\,t)=\varphi_2(t),\ t>0;\ [\partial T(x,\,t)/\partial x]_{x=y(t)}=\Psi(t),\ t>0. \quad (26)$$

Problems of this kind arise in water filtration studies (see references in [3, 4]); a particular case is the Stefan problem with $\Phi_0(x)=\text{const}\geq 0$, $\varphi_2(t)=0$; $\Psi=A\,dy/dt$, where $A=L\rho/\lambda$. In problem (24)-(26) we are required to find functions $T(x,\,t)$ and $y(t)$, such that $T(x,\,t)\in C^2(\Omega_t)$, $\text{grad}_x T(x,\,t)\in C^0(\overline{\Omega}_t)$; the function $y(t)$ is differentiable for $t\geq 0$, $dy/dt\geq\varphi_2'(t)/\Psi(t)$; $y(0)=y_0>0$. This problem does not always have a solution. For example, when $\Phi_0(x)={}_1(t)=\varphi_2(t)=\text{const}$, $\Psi(t)\neq 0$, there are no solutions. Let us write the solution of problem (24)-(26) using the relations (18) and (22):

$$\begin{aligned} T(x,\,t)=&\frac{2}{y(t)}\sum_{n=1}^{\infty}\exp\left[\left(-\left(\frac{n\pi\sqrt{a}}{y(t)}\right)^2 t\right]\sin\frac{n\pi x}{y(t)}\int\limits_0^{y_0}\Phi_0(x)\sin\frac{n\pi x}{y(t)}dx+\right.\\ &+\frac{2a}{y(t)}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{y(t)}\int\limits_0^t\exp\left[-\left(\frac{n\pi\sqrt{a}}{y(t)}\right)^2(t-\tau)\right]\Psi(\tau)\sin\frac{n\pi y(\tau)}{y(t)}\,d\tau+\\ &+\frac{2\pi a}{y^2(t)}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{y(t)}\int\limits_0^t\exp\left[-\left(\frac{n\pi\sqrt{a}}{y(t)}\right)^2(t-\tau)\right]\left\{n\varphi_1(\tau)+\right. \end{aligned} \quad (27)$$

$$+\left[\frac{dy(\tau)}{d\tau}\sin\frac{n\pi y(\tau)}{y(t)}-\frac{n\pi a}{y(t)}\cos\frac{n\pi y(\tau)}{y(t)}\right]\varphi_2(\tau)\,d\tau=$$

$$=[x/y(t)]\varphi_2(t)+[1-x/y(t)]\varphi_1(t)+\frac{2}{y(t)}\sum_{n=1}^{\infty}\exp\left[-\left(\frac{n\pi\sqrt{a}}{y(t)}\right)^2 t\right]\times$$

$$\times\sin\frac{n\pi x}{y(t)}\int_0^{y_0}\Phi_1(x)\sin\frac{n\pi x}{y(t)}\,dx+\frac{2a}{y(t)}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{y(t)}\times$$

$$\times\int_0^t\exp\left[-\left(\frac{n\pi\sqrt{a}}{y(t)}\right)^2(t-\tau)\right]q(\tau)\sin\frac{n\pi y(\tau)}{y(t)}\,d\tau+$$

$$+\frac{2}{y(t)}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{y(t)}\int_0^t\int_0^{y(\tau)}\exp\left[-\left(\frac{n\pi\sqrt{a}}{y(t)}\right)^2(t-\tau)\right]F(x,\tau)\sin\frac{n\pi y(\tau)}{y(t)}\,d\tau dx.$$

Here $\Phi_1(x) = \Phi_0(x) - (x/y_0)\varphi_2(0) + (1 - x/y_0)\varphi_1(0)$; $q(t) = \Psi(t) - [\varphi_2(t) - \varphi_1(t)]/y(t)$; $F(x, t) = -\partial/\partial t\{[x/y(t)]\varphi_2(t) + [1 - x/y(t)]\varphi_1(t)\}$.

A second form of the solution (27) may also be obtained from the relations (18) and (22) if through a simple substitution we first eliminate nonhomogeneities in the boundary conditions for T(0, t) and T(y(t), t); this yields a solution in the form of the series (18), converging absolutely and uniformly up to the boundary Ω_t [3]; moreover, $T(x, t) \in C^2(\Omega_t) \cap C^0(\bar{\Omega}_t)$, if $\Phi_1(0) = \Phi_1(y_0) = 0$, $q(0) = \Phi_1'(y_0)$. This form is more suitable for numerical calculations since the series appearing in it have rapid convergence [1, 2]. The unknown function y(t) in (24)-(26) is determined from the functional equation

$$\int_0^{\infty}\exp(-ap\tau)\varphi_2(\tau)\operatorname{ch}[y(\tau)\sqrt{p}]\,d\tau-\int_0^{\infty}\exp(-ap\tau)\Psi(\tau)\operatorname{sh}[y(\tau)\sqrt{p}]\,d\tau=$$
$$=\sqrt{p}\int_0^{\infty}\exp(-ap\tau)\varphi_1(\tau)\,d\tau+(1/a)\int_0^{y_0}\Phi_0(x)\operatorname{sh}(x\sqrt{p})\,dx, \tag{28}$$

obtained from Eq. (17) by the approach used in [6] (where a scheme is given for obtaining an asymptotic solution of Eq. (28)).

The expressions (27) and (28) possess sufficient generality, describing analytical solutions of the problem (24)-(26) both for the degenerate as well as for the nondegenerate cases. Indeed, on the basis of Eqs. (27) and (28), we obtain a solution of the classical Stefan problem for the degenerate case:

$$\partial T/\partial t=a\partial^2T/\partial x^2,\ 0<x<y(t),\ t>0;\ y(0)=0; \tag{29}$$

$$T(0,\ t)=\varphi_0=\text{const}<0,\ t>0; \tag{30}$$

$$T[y(t),\ t]=0,\ t>0; \tag{31}$$

$$[\partial T(x,\ t)/\partial x]_{x=y(t)}=A\,dy/dt,\ t>0\ (A=L\rho/\lambda), \tag{32}$$

being also in two distinct (equivalent) functional forms, using at first the approach (15) and then relations (18) and (22). First of all, we introduce relations useful in applications (obtained through successive integration by parts):

$$\int_0^y\exp(-x^2)x^{2k}dx=\frac{(2k)!}{k!}y^{2k}\exp(-y^2)\sum_{m=0}^{\infty}\frac{2^{2m}(k+m)!\,y^{2m+1}}{[2(k+m)+1]!}; \tag{33}$$

$$\int_0^y\exp(x^2)x^{2k+1}dx=\frac{(-1)^{k+1}k!}{2}\exp(y^2)\sum_{m=k+1}^{\infty}\frac{(-1)^m y^{2m}}{m!}. \tag{34}$$

Putting k = 0 in Eq. (33), we obtain yet another necessary relationship

$$\sqrt{a\pi}\exp(\beta^2/4a)\Phi(\beta/2\sqrt{a})=\sum_{m=0}^{\infty}\frac{\beta^{2m+1}m!}{a^m(2m+1)!}, \tag{35}$$

where $\Phi(z) = (2/\sqrt{\pi}) \int\limits_0^z \exp(-z^2)\,dz$ is the Laplace function. Further, using Eq. (15), we may write T(x, t) in the form

$$T(x, t) = -A \sum_{n=0}^{\infty} \frac{1}{a^n (2n+1)!} \frac{\partial^n}{\partial t^n} \left\{ [y(t) - x]^{2n+1} \frac{dy}{dt} \right\}, \tag{36}$$

satisfying Eq. (29) and the conditions (31) and (32). The condition (30) leads to the relation

$$\sum_{n=0}^{\infty} \frac{1}{a^n (2n+1)!} \frac{d^n}{dt^n} \left[y^{2n+1}(t) \frac{dy}{dt} \right] = -(\varphi_0/A) = \text{const}, \tag{37}$$

which is satisfied for all $t > 0$ if $y^{2n+1}(t)dy/dt = \gamma t^n$, which leads to a law of motion of the boundary in the form $y = \beta\sqrt{t}$, where the coefficient β is to be determined. To this end, in Eq. (37) we put $y = \beta\sqrt{t}$ and then sum. The result is a functional equation in β:

$$-2\varphi_0/A\sqrt{a\pi} = \beta \exp(\beta^2/4a)\Phi(\beta/2\sqrt{a}), \tag{38}$$

which was considered in [8]. The desired solution, of self-similar form, is obtained from Eq. (36) with $y(t) = \beta\sqrt{t}$ upon taking Eqs. (33)-(35) into account:

$$T\left(\frac{x}{2\sqrt{at}}\right) = -A \sum_{n=0}^{\infty} \frac{1}{a^n (2n+1)!} \frac{d^n}{dt^n} \left[(\beta\sqrt{t} - x)^{2n+1} \frac{\beta}{2\sqrt{t}} \right] =$$

$$= -A \sum_{n=0}^{\infty} \frac{\beta^{2(n+1)}}{a^n [2(n+1)]!} \frac{d^n}{dt^{n+1}} \left[(\sqrt{t})^{2(n+1)} \left(1 - \frac{x}{\beta\sqrt{t}}\right)^{2(n+1)} \right] =$$

$$= -A \sum_{n=0}^{\infty} \frac{\beta^{2(n+1)}(n+1)!}{a^n [2(n+1)]!} + A \sum_{n=0}^{\infty} \frac{\beta^{2(n+1)}}{2^{n+1} a^n} \sum_{m=0}^{n} \frac{(x/\beta\sqrt{t})^{2m+1} \prod\limits_{k=0}^{n} (2n - 2m - 2k + 1)}{(2m+1)!\,(2n-2m+1)!} =$$

$$= -\frac{A\beta\sqrt{a\pi}}{2} \exp(\beta^2/4a)\, \Phi\left(\frac{\beta}{2\sqrt{a}}\right) + \frac{A\beta}{2} \exp\left(\frac{\beta^2}{4a}\right) \times$$

$$\times \sum_{n=0}^{\infty} \frac{(-1)^n (x/\sqrt{t})^{2n}}{4^n a^n n!} \sum_{n=0}^{\infty} \frac{n!\,(x/\sqrt{t})^{2n+1}}{a^n (2n+1)!} = \frac{A\beta\sqrt{a\pi}}{2} \exp\left(\frac{\beta^2}{4a}\right) \times$$

$$\times \left[\Phi\left(\frac{\beta}{2\sqrt{a}}\right) - \Phi\left(\frac{x}{2\sqrt{at}}\right) \right] = \varphi_0 + A\beta\sqrt{a\pi}/2 \exp\left(\frac{\beta^2}{4a}\right) \Phi\left(\frac{x}{2\sqrt{at}}\right). \tag{39}$$

With the aid of relations (27) and (28) we obtain yet another analytical form of the solution of problem (29)-(32), different from Eq. (39). If in Eq. (28) (for $y_0 = 0$, $\Phi_0(x) = 0$) we take into account the form of the boundary conditions in Eqs. (30)-(32), we obtain a functional equation for y(t):

$$p \int\limits_0^{\infty} \exp(-ap\tau)\, \text{ch}\,[y(\tau)\sqrt{p}]\, d\tau = (1/a)(1 - \varphi_0/aA) = \text{const}.$$

The latter holds for $y(\tau) = \beta\sqrt{\tau}$, resulting in a relationship for β:

$$2 \int\limits_0^{\infty} z \exp(-az^2)\, \text{ch}(\beta z)\, dz = (1/a)(1 - \varphi_0/aA),$$

which yields the expression (38) upon evaluating the integral. We obtain the desired solution from Eq. (27) with the form of the boundary conditions taken into account:

$$T(x, t) = \frac{2a\beta A}{\pi} \sum_{n=1}^{\infty} \left\{ \frac{1}{n} \exp\left[-\left(\frac{n\pi\sqrt{a}}{\beta} \right)^2 \right] \int\limits_0^{n\pi/\beta} \exp(az^2) \sin\beta z\, dz + \right.$$

$$\left. + \frac{2\varphi_0}{n\pi} \left[1 - \exp\left(-\left(\frac{n\pi\sqrt{a}}{\beta} \right)^2 \right) \right] \right\} \sin\frac{n\pi x}{\beta\sqrt{t}} = \sum_{n=1}^{\infty} a_n' \sin\frac{n\pi x}{\beta\sqrt{t}}. \tag{40}$$

The coefficients a_n' in Eq. (40) can be transformed with the aid of relations (33)-(35) if we expand in a series involving sin βz:

$$a_n' = \frac{2a\beta A}{n\pi}\exp\left(-\frac{n^2\pi^2 a}{\beta^2}\right)\sum_{k=0}^{\infty}\frac{(-1)^k\beta^{2k+1}}{(2k+1)!}\int_0^{n\pi/\beta}\exp(az^2)z^{2k+1}dz +$$

$$+\frac{\beta A}{n\pi}\sqrt{a\pi}\exp\left(\frac{\beta^2}{4a}\right)\Phi\left(\frac{\beta}{2\sqrt{a}}\right)\left[\exp\left(-\frac{n^2\pi^2 a}{\beta^2}\right)-1\right]=$$

$$=\frac{A\beta^2}{n\pi}\left\{-\sum_{k=0}^{\infty}\frac{\beta^{2k}k!}{a^k(2k+1)!}\sum_{m=k+1}^{\infty}\frac{(-1)^m(an^2\pi^2/\beta^2)^m}{m!}+\sum_{n=0}^{\infty}\frac{\beta^{2k}k!}{a^k(2k+1)!}\sum_{m=1}^{\infty}\frac{(-1)^m(an^2\pi^2/\beta^2)^m}{m!}\right\}=$$

$$=\frac{A\beta^2}{n\pi}\left\{-\sum_{k=0}^{\infty}\frac{\beta^{2k}k!}{a^k(2k+1)!}+\sum_{k=0}^{\infty}\frac{\beta^{2k}k!}{a^k(2k+1)!}\sum_{m=0}^{k}\frac{(-1)^m(an^2\pi^2/\beta^2)^m}{m!}\right\}=$$

$$=\frac{A\beta^2}{n\pi}\left\{-\sum_{k=0}^{\infty}\frac{\beta^{2k}k!}{a^k(2k+1)!}+\sum_{k=0}^{\infty}\frac{(-1)^k(an^2\pi^2/\beta^2)^k}{k!}\times\right.$$

$$\left.\times\sum_{m=k}^{\infty}\frac{\beta^{2m}m!}{a^m(2m+1)!}\right\}=\frac{A\beta^2}{n\pi}\sum_{k=1}^{\infty}\frac{(-1)^k(an^2\pi^2/\beta^2)^k}{k!}\sum_{m=k}^{\infty}\frac{\beta^{2m}m!}{a^m(2m+1)!}.$$

Thus, the solution of problem (29)-(32) has the form

$$T(x,t)=\sum_{n=1}^{\infty}a_n'\sin\frac{n\pi x}{\beta\sqrt{t}},$$

$$a_n'=\frac{A\beta^2}{n\pi}\sum_{k=1}^{\infty}\frac{(-1)^k(an^2\pi^2/\beta^2)^k}{k!}\sum_{m=k}^{\infty}\frac{\beta^{2m}m!}{a^m(2m+1)!}, \tag{41}$$

where β is a solution of the transcendental equation (38). The second functional form of the solution of problem (29)-(32) is the expression (39).

We show now that these two forms are equivalent (taking note of the fact that an immediate testing of relation (41) as to the realizability of conditions (29)-(32) does not yield the obvious result). To this end, we write Eq. (39) in the form of the trigonometric series

$$T(x,t)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{\beta\sqrt{t}},\quad 0<x<\beta\sqrt{t}, \tag{42}$$

from whence we have

$$b_n=(2/\beta\sqrt{t})\int_0^{\beta\sqrt{t}}\left[\varphi_0+\frac{A\beta}{2}\sqrt{a\pi}\exp\left(\frac{\beta^2}{4a}\right)\Phi\left(\frac{x}{2\sqrt{at}}\right)\right]\sin\frac{n\pi x}{\beta\sqrt{t}}dx=$$

$$=\frac{2\varphi_0}{n\pi}(1-\cos n\pi)+\frac{A\beta}{n\pi}\sqrt{a\pi}\exp\left(\frac{\beta^2}{4a}\right)\Phi\left(\frac{\beta}{2\sqrt{a}}\right)(1-\cos n\pi)+$$

$$+\frac{A\beta}{n\pi}\exp\left(\frac{\beta^2}{4a}\right)\sum_{k=1}^{\infty}\frac{(-1)^k(n\pi)^{2k}}{(2k)!}\int_0^1\exp\left(-\frac{\beta^2y^2}{4a}\right)y^{2k}dy=$$

$$=\frac{A\beta^2}{n\pi}\sum_{k=1}^{\infty}\frac{(-1)^k(an^2\pi^2/\beta^2)^k}{k!}\sum_{m=k}^{\infty}\frac{\beta^{2m}m!}{a^m(2m+1)!},$$

i.e., $a_n' = b_n$. Thus, solution (39) can be represented in the equivalent analytical form (41) corresponding to the approach employed, either that of Eq. (15) or that of relations (18) and (22).

It is also of interest to write the analytical solution to the thermal problem of classical type:

$$\partial T/\partial t = a\partial^2 T/\partial x^2 + F(x, t), \quad 0 < x < y(t), \quad t > 0; \tag{43}$$

$$T(x, 0) = \Phi_0(x), \quad 0 < x < y(0) = y_0 > 0; \tag{44}$$

$$T(0, t) = T(y(t), t) = 0, \quad t > 0, \tag{45}$$

this time, however, in a domain with a moving boundary, where y(t) describes displacement of the boundary. The presence of homogeneous boundary conditions does not restrict the generality of the problem since, with the aid of simple substitutions, one can always transfer nonhomogeneities from boundary conditions into an initial condition and into the heat-conduction equation [3]. In the case of a degenerate domain (y_0 = 0) an exact solution of the problem (43)-(45) was given in [3] on the basis of Eq. (15). For the nondegenerate case this remains an open question; from relations (18) and (22) we have

$$\begin{aligned} T(x, t) &= \frac{2}{y(t)} \sum_{n=1}^{\infty} \exp\left[-\left(\frac{n\pi\sqrt{a}}{y(t)}\right)^2 t\right] \sin\frac{n\pi x}{y(t)} \int_0^{y_0} \Phi_0(x') \sin\frac{n\pi x'}{y(t)} dx' + \\ &+ \frac{2}{y(t)} \sum_{n=1}^{\infty} \sin\frac{n\pi x}{y(t)} \left[\int_0^t \int_0^{y(\tau)} \exp\left[-\left(\frac{\sqrt{a}n\pi}{y(t)}\right)^2 (t-\tau)\right] F(x', \tau) \sin\frac{n\pi x'}{y(t)} d\tau dx' + \right. \\ &\left. + a \int_0^t \exp\left[-\left(\frac{\sqrt{a}n\pi}{y(t)}\right)^2 (t-\tau)\right] q(\tau) \sin\frac{n\pi y(\tau)}{y(t)} d\tau\right] = \\ &= \frac{1}{2y(t)} \int_0^{y_0} \Phi_0(x') \left[\vartheta\left(\frac{x-x'}{2y(t)}, \frac{at}{y^2(t)}\right) - \vartheta\left(\frac{x+x'}{2y(t)}, \frac{at}{y^2(t)}\right)\right] dx' + \\ &+ \frac{1}{2y(t)} \int_0^t \int_0^{y(\tau)} F(x', \tau) \left[\vartheta\left(\frac{x-x'}{2y(t)}, \frac{a(t-\tau)}{y^2(t)}\right) - \vartheta\left(\frac{x+x'}{2y(t)}, \right.\right. \\ &\left.\left. \frac{a(t-\tau)}{y^2(t)}\right)\right] d\tau dx' + \frac{a}{2y(t)} \int_0^t q(\tau) \left[\vartheta\left(\frac{x-y(\tau)}{2y(t)}, \frac{a(t-\tau)}{y^2(t)}\right) - \vartheta\left(\frac{x+y(\tau)}{2y(t)}, \frac{a(t-\tau)}{y^2(t)}\right)\right] d\tau, \end{aligned} \tag{46}$$

if we introduce the Jacobi theta-function:

$$\vartheta(x, t) = 1 + 2\sum_{k=1}^{\infty} \exp(-k^2\pi^2 t)\cos(2k\pi x) = \frac{1}{\sqrt{\pi t}} \sum_{k=-\infty}^{k=+\infty} \exp\left(-\frac{(x-k)^2}{4t}\right).$$

The expression (46) contains the unknown thermal flow function $q(t) = (\partial T/\partial x)_{x=y(t)}$; using the relation found for T(x, t), we arrive at an integral equation for q(t):

$$q(t) = \varphi(t) + \frac{a}{2y(t)} \int_0^t K(t, \tau) q(\tau) d\tau, \tag{47}$$

where φ(t) and K(t, τ) are known functions (which are easily written out from Eq. (46)). The solution of Eq. (47) can be obtained in the usual way through use of the Picard process in the form

$$q(t) = \sum_{n=0}^{\infty} \left(\frac{a}{2y(t)}\right)^n q_n(t); \tag{48}$$

$$q_0(t) = \varphi(t); \quad q_n(t) = \int_0^t K(t, \tau) q_{n-1}(\tau) d\tau, \quad n \geq 1. \tag{49}$$

In an analogous manner we can consider a radial temperature field T(r, t) in the region r ∈ [0, R(t)], t > 0, being also for problems with phase transitions (Stefan type) and for classical problems (where the functions R(t) is specified). These cases remain to be treated.

NOTATION

$y(t)$, function describing displacement of the boundary; $y(0) = y_0$, initial position of the boundary; x, spatial coordinate; t, time; L, heat of transition; ρ, density; λ, thermal conductivity; a, thermal diffusivity.

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